A HEAT-CONDUCTION PROBLEM

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A heat-conduction problem is posed with a branching of the heat flux at the boundaries, leading to a system of Volterra integral equations of the second kind; the system is transformed to a form which can be solved by the operator series method.

We consider the problem of the temperature distribution along two linear heat conductors with thermally insulated lateral surfaces. We assume that the heat flux through each end cross section consists of two terms, one of which is proportional to the temperature difference between the corresponding ends of the heat conductors, and the second of which is proportional to the temperature difference between the end of the heat conductor and a body whose temperature varies with time in a specified way. This problem is reduced to solving the following system [1]:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial u}{\partial t}, \quad 0 \leqslant x \leqslant l_1;$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{1}{b^2}\frac{\partial v}{\partial t}, \quad 0 \leqslant y \leqslant l_2;$$

$$u|_{t=-\infty} = 0; \qquad v|_{t=-\infty} = 0; \tag{1}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} - \alpha_1 u|_{x=0} + \beta_1 v|_{y=0} = \Phi_1(t), \tag{2}$$

$$\left.\frac{\partial v}{\partial y}\right|_{y=0} - \alpha_2 v|_{y=0} + \beta_2 u|_{x=0} = \Phi_2(t), \tag{3}$$

$$-\left.\frac{\partial u}{\partial x}\right|_{x=l_1} - \alpha_3 u|_{x=l_1} + \beta_3 v|_{y=l_2} = \Phi_3(t), \tag{4}$$

$$-\left.\frac{\partial v}{\partial y}\right|_{y=l_2} - \alpha_4 v|_{y=l_2} + \beta_4 u|_{x=l_1} = \Phi_4(t), \tag{5}$$

where α_k and β_k (k = 1, 2, 3, 4) are given positive constants, and the $\Phi_k(t)$ (k = 1, 2, 3, 4) are given functions of the time satisfying the condition

$$\Phi_k(t)\cdot\exp\left(-\frac{t}{2}\right) \in L_2(-\infty, \infty), \quad k = 1, 2, 3, 4. \tag{6}$$

Reduction to a System of Integral Equations. We seek the solution of the system in terms of the thermal potentials of a single layer:

$$u(x, t) = \frac{a}{2\sqrt{\pi}}\int_{-\infty}^{t}(t-\tau)^{-\frac{1}{2}}\left[\rho_1(\tau)\exp\left(-\frac{x^2}{4a^2(t-\tau)}\right) + \rho_2(\tau)\exp\left(-\frac{(x-l_1)^2}{4a^2(t-\tau)}\right)\right]d\tau, \tag{7}$$

$$v(y, t) = \frac{b}{2\sqrt{\pi}}\int_{-\infty}^{t}(t-\tau)^{-\frac{1}{2}}\left[\rho_3(\tau)\exp\left(-\frac{y^2}{4b^2(t-\tau)}\right) + \rho_4(\tau)\exp\left(-\frac{(y-l_2)^2}{4b^2(t-\tau)}\right)\right]d\tau. \tag{8}$$

It follows from (7) and (8) that conditions (1) are satisfied automatically. The problem consists in finding the unknown strengths $\rho_k(t)$ (k = 1, 2, 3, 4) so as to satisfy boundary conditions (2)-(5). Expressing these boundary conditions by using (7) and (8) we obtain four integral equations in four unknown

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functions, which after the reduction of like terms take the form

$$-a\alpha_1 \int_{-\infty}^{t} \frac{1}{\sqrt{\pi(t-\tau)}} \rho_1(\tau)\, d\tau + \int_{-\infty}^{t} \left[\frac{l_1}{a\sqrt{\pi(t-\tau)^3}} - \alpha_1 \frac{a}{\sqrt{\pi(t-\tau)}}\right] \times$$

$$\times \exp\left(-\frac{l_1^2}{4a^2(t-\tau)}\right) \rho_2(\tau)\, d\tau + b\beta_1 \int_{-\infty}^{t} \frac{1}{\sqrt{\pi(t-\tau)}} \rho_3(\tau)\, d\tau +$$

$$+ b\beta_1 \int_{-\infty}^{t} \frac{1}{\sqrt{\pi(t-\tau)}} \exp\left(-\frac{l_2^2}{4b^2(t-\tau)}\right) \rho_4(\tau)\, d\tau - \rho_1(t) = 2\Phi_1(t), \tag{9}$$

$$a\beta_2 \int_{-\infty}^{t} \frac{1}{\sqrt{\pi(t-\tau)}} \rho_1(\tau)\, d\tau + a\beta_2 \int_{-\infty}^{t} \frac{1}{\sqrt{\pi(t-\tau)}} \exp\left(-\frac{l_1^2}{4a^2(t-\tau)}\right) \rho_2(\tau)\, d\tau -$$

$$-b\alpha_2 \int_{-\infty}^{t} \frac{1}{\sqrt{\pi(t-\tau)}} \rho_3(\tau)\, b\tau + \int_{-\infty}^{t} \left[\frac{l_2}{b\sqrt{\pi(t-\tau)^3}} - \alpha_2 \frac{b}{\sqrt{\pi(t-\tau)}}\right] \times$$

$$\times \exp\left(-\frac{l_2^2}{4b^2(t-\tau)}\right) \rho_4(\tau)\, d\tau - \rho_3(t) = 2\Phi_2(t), \tag{10}$$

$$\int_{-\infty}^{t} \left[\frac{l_1}{a\sqrt{\pi(t-\tau)^3}} - \alpha_3 \frac{a}{\sqrt{\pi(t-\tau)}}\right] \exp\left(-\frac{l_1^2}{4a^2(t-\tau)}\right) \rho_1(\tau)\, d\tau -$$

$$-a\alpha_3 \int_{-\infty}^{t} \frac{1}{\sqrt{\pi(t-\tau)}} \rho_2(\tau)\, d\tau + b\beta_3 \int_{-\infty}^{t} \frac{1}{\sqrt{\pi(t-\tau)}} \exp\left(-\frac{l_2^2}{4b^2(t-\tau)}\right) \times$$

$$\times \rho_3(\tau)\, d\tau + b\beta_3 \int_{-\infty}^{t} \frac{1}{\sqrt{\pi(t-\tau)}} \rho_4(\tau)\, d\tau - \rho_2(t) = 2\Phi_3(t), \tag{11}$$

$$a\beta_4 \int_{-\infty}^{t} \frac{1}{\sqrt{\pi(t-\tau)}} \exp\left(-\frac{l_1^2}{4a^2(t-\tau)}\right) \rho_1(\tau)\, d\tau +$$

$$+ a\beta_4 \int_{-\infty}^{t} \frac{1}{\sqrt{\pi(t-\tau)}} \rho_2(\tau)\, d\tau + \int_{-\infty}^{t} \left[\frac{l_2}{b\sqrt{\pi(t-\tau)^3}} - \alpha_4 \frac{b}{\sqrt{\pi(t-\tau)}}\right] \times$$

$$\times \exp\left(-\frac{l_2^2}{4b^2(t-\tau)}\right) \rho_3(\tau)\, d\tau - b\alpha_4 \int_{-\infty}^{t} \frac{1}{\sqrt{\pi(t-\tau)}} \rho_4(\tau)\, d\tau -$$

$$-\rho_4(t) = 2\Phi_4(t). \tag{12}$$

Transformation of Kernels of the System of Integral Equations. The difference kernels of the integral operators appearing in Eqs. (9)-(12) do not permit a direct solution of this system by the operator series method. Therefore, we first transform system (9)-(12) so as to obtain kernels depending on the products of arguments. To do this we introduce new variables s and σ related to t and τ by the equations

$$t = \ln s, \qquad \tau = -\ln \sigma. \tag{13}$$

In addition, we introduce new unknown functions $\varphi_k(s)$:

$$\varphi_k(s) = \rho_k(-\ln s), \qquad k = 1, 2, 3, 4. \tag{14}$$

Furthermore, we introduce the notation

$$h_k(s) = 2\Phi_k(-\ln s), \qquad k = 1, 2, 3, 4. \tag{15}$$

Now system (9)-(12) takes the form

$$-a\alpha_1 \int_{\frac{1}{s}}^{\infty} \frac{1}{\sqrt{\pi}\, s\sigma \sqrt{\ln s\sigma}} \varphi_1(\sigma)\, d\sigma + \int_{\frac{1}{s}}^{\infty} \left[\frac{l_1}{a\sqrt{\pi}\, s\sigma \sqrt{\ln^3 s\sigma}} - \right.$$

$$-\alpha_1 \frac{a}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\Big]\exp\left(-\frac{l_1^2}{4a^2\ln s\sigma}\right)\varphi_2(\sigma)\,d\sigma+$$

$$+b\beta_1\int_{\frac{1}{s}}^{\infty}\frac{1}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\varphi_3(\sigma)\,d\sigma+b\beta_1\int_{\frac{1}{s}}^{\infty}\frac{1}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\times$$

$$\times\exp\left(-\frac{l_2^2}{4b^2\ln s\sigma}\right)\varphi_4(\sigma)\,d\sigma-\frac{1}{s}\varphi_1\left(\frac{1}{s}\right)=\frac{1}{s}h_1\left(\frac{1}{s}\right), \tag{16}$$

$$a\beta_1\int_{\frac{1}{s}}^{\infty}\frac{1}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\varphi_1(\sigma)\,d\sigma+a\beta_2\int_{\frac{1}{s}}^{\infty}\frac{1}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\times$$

$$\times\exp\left(-\frac{l_1^2}{4a^2\ln s\sigma}\right)\varphi_2(\sigma)\,d\sigma-b\alpha_2\int_{\frac{1}{s}}^{\infty}\frac{1}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\varphi_3(\sigma)\,d\sigma+$$

$$+\int_{\frac{1}{s}}^{\infty}\left[\frac{l_2}{b\sqrt{\pi s\sigma}\sqrt{\ln^3 s\sigma}}-\alpha_2\frac{b}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\right]\exp\left(-\frac{l_2^2}{4b^2\ln s\sigma}\right)\varphi_4(\sigma)\,d\sigma-$$

$$-\frac{1}{s}\varphi_3\left(\frac{1}{s}\right)=\frac{1}{s}h_2\left(\frac{1}{s}\right), \tag{17}$$

$$\int_{\frac{1}{s}}^{\infty}\left[\frac{l_1}{a\sqrt{\pi s\sigma}\sqrt{\ln^3 s\sigma}}-\alpha_3\frac{a}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\right]\times$$

$$\times\exp\left(-\frac{l_1^2}{4a^2\ln s\sigma}\right)\varphi_1(\sigma)\,d\sigma-a\alpha_3\int_{\frac{1}{s}}^{\infty}\frac{1}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\varphi_2(\sigma)\,d\sigma+$$

$$+b\beta_3\int_{\frac{1}{s}}^{\infty}\frac{1}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\exp\left(-\frac{l_2^2}{4b^2\ln s\sigma}\right)\varphi_3(\sigma)\,d\sigma+$$

$$+b\beta_3\int_{\frac{1}{s}}^{\infty}\frac{1}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\varphi_4(\sigma)\,d\sigma-\frac{1}{s}\varphi_2\left(\frac{1}{s}\right)=\frac{1}{s}h_3\left(\frac{1}{s}\right), \tag{18}$$

$$a\beta_4\int_{\frac{1}{s}}^{\infty}\frac{1}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\exp\left(-\frac{l_1^2}{4a^2\ln s\sigma}\right)\varphi_1(\sigma)\,d\sigma+$$

$$+a\beta_4\int_{\frac{1}{s}}^{\infty}\frac{1}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\varphi_2(\sigma)\,d\sigma+\int_{\frac{1}{s}}^{\infty}\Big[\frac{l_2}{b\sqrt{\pi s\sigma}\sqrt{\ln^3 s\sigma}}-$$

$$-\alpha_4\frac{b}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\Big]\exp\left(-\frac{l_2^2}{4b^2\ln s\sigma}\right)\varphi_3(\sigma)\,d\sigma-$$

$$-b\alpha_4\int_{\frac{1}{s}}^{\infty}\frac{1}{\sqrt{\pi s\sigma}\sqrt{\ln s\sigma}}\varphi_4(\sigma)\,d\sigma-\frac{1}{s}\varphi_4\left(\frac{1}{s}\right)=\frac{1}{s}h_4\left(\frac{1}{s}\right). \tag{19}$$

Transformation of the Integral Operators of the System. The integral operators in system (16)-(19) have the form

$$\int_{\frac{1}{s}}^{\infty}K(s\sigma)\,\varphi(\sigma)\,d\sigma. \tag{20}$$

We transform each integral operator (20) to the form

$$\frac{d}{ds}\left\{s\int_{\frac{1}{s}}^{\infty}\tilde{K}(s\sigma)\varphi(\sigma)\,d\sigma\right\}, \tag{21}$$

where we call the function $\tilde{K}(x)$ the kernel of the integral operator (21). We find the kernels of the operators (21) by the formula

$$\tilde{K}(x)=\frac{1}{x}\int_{1}^{x}K(t)\,dt. \tag{22}$$

The function K(x) for the integral operators which appear in Eqs. (16)-(19) has one of the following five forms:

$$K_1(x)=\frac{1}{\sqrt{\pi}\,x\sqrt{\ln x}};$$

$$K_2(x)=\frac{a}{\sqrt{\pi}\,x\sqrt{\ln x}}\exp\left(-\frac{l_1^2}{4a^2\ln x}\right);$$

$$K_3(x)=\frac{l_1}{a\sqrt{\pi}\,x\sqrt{\ln^3 x}}\exp\left(-\frac{l_1^2}{4a^2\ln x}\right);$$

$$K_4(x)=\frac{b}{\sqrt{\pi}\,x\sqrt{\ln x}}\exp\left(-\frac{l_2^2}{4b^2\ln x}\right);$$

$$K_5(x)=\frac{l_2}{b\sqrt{\pi}\,x\sqrt{\ln^3 x}}\exp\left(-\frac{l_2^2}{4b^2\ln x}\right).$$

We find the corresponding kernels of the integral operators considered, reduced to the form (21):

$$\tilde{K}_1(x)=\frac{2\sqrt{\ln x}}{\sqrt{\pi}\,x}; \tag{23}$$

$$\tilde{K}_2(x)=\frac{2a\sqrt{\ln x}}{\sqrt{\pi}\,x}\exp\left(-\frac{l_1^2}{4a^2\ln x}\right)-\frac{l_1}{x}\operatorname{erfc}\left(\frac{l_1}{2a\sqrt{\ln x}}\right); \tag{24}$$

$$\tilde{K}_3(x)=\frac{2}{x}\operatorname{erfc}\left(\frac{l_1}{2a\sqrt{\ln x}}\right); \tag{25}$$

$$\tilde{K}_4(x)=\frac{2b\sqrt{\ln x}}{\sqrt{\pi}\,x}\exp\left(-\frac{l_2^2}{4b^2\ln x}\right)-\frac{l_2}{x}\operatorname{erfc}\left(\frac{l_2}{2b\sqrt{\ln x}}\right); \tag{26}$$

$$\tilde{K}_5(x)=\frac{2}{x}\operatorname{erfc}\left(\frac{l_2}{2b\sqrt{\ln x}}\right). \tag{27}$$

In Eqs. (24)-(27) we have used the customary notation

$$\operatorname{erfc}x=\frac{2}{\sqrt{\pi}}\int_{x}^{\infty}e^{-t^2}\,dt. \tag{28}$$

We denote integral operators of the form (21) with kernels $\tilde{K}_i$ (i = 1, 2, 3, 4, 5) by the symbols V_i (i = 1, 2, 3, 4, 5). In addition, we introduce the linear operator S defined by the expression

$$S\varphi(x)=\frac{1}{x}\varphi\left(\frac{1}{x}\right). \tag{29}$$

Then the system of integral equations (16)-(19) can be written in the following operator form:

$$\begin{aligned}
&-(a\alpha_1V_1+S)\varphi_1(s)+(V_3-\alpha_1V_2)\varphi_2(s)+b\beta_1V_1\varphi_3(s)+\beta_1V_4\varphi_4(s)=Sh_1(s),\\
&a\beta_1V_1\varphi_1(s)+\beta_2V_2\varphi_2(s)-(b\alpha_2V_1+S)\varphi_3(s)+(V_5-\alpha_2V_4)\varphi_4(s)=Sh_2(s),\\
&(V_3-\alpha_3V_2)\varphi_1(s)-(a\alpha_3V_1+S)\varphi_2(s)+\beta_3V_4\varphi_3(s)+b\beta_3V_1\varphi_4(s)=Sh_3(s),\\
&\beta_4V_2\varphi_1(s)+a\beta_4V_1\varphi_2(s)+(V_5-\alpha_4V_4)\varphi_3(s)-(b\alpha_4V_1+S)\varphi_4(s)=Sh_4(s).
\end{aligned} \tag{30}$$

The system of integral equations can now be solved by the operator series method.

NOTATION

u, v — are the temperatures of the first and second linear heat conductors;
a^2, b^2 — are the thermal diffusivities of the first and second heat conductors;

l_1, l_2	are the lengths of the first and second heat conductors;
α_k, β_k (k = 1, 2, 3, 4)	are the constant coefficients in the boundary conditions of the system;
$\Phi_k(t)$ (k = 1, 2, 3, 4)	are the given functions in the boundary conditions of the system;
ρ_k (k = 1, 2, 3, 4)	are the unknown strengths of the thermal potentials;
K_i (i = 1, 2, 3, 4, 5)	are the kernels of the integral operators;
V_i (i = 1, 2, 3, 4, 5), S	are the linear integral operators.

LITERATURE CITED

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